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Monge–Ampère equations and generalized complex geometry— The two-dimensional case

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Abstract

We associate an integrable generalized complex structure with each two-dimensional symplectic Monge–Ampère equation of divergent type and, using the Gualtieri $\overline{\partial}$ operator, we characterize the conservation laws and the generating functions of such an equation as generalized holomorphic objects.
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0. Introduction

A general approach to the study of non-linear partial differential equations, which goes back to Sophus Lie, is to see a kth-order equation on an n-dimensional manifold N^n as a closed subset in the manifold of k-jets J^kN. In particular, a second-order differential equation lives in the space J^2N. Nevertheless, as was noticed by Lychagin in his seminal paper "Contact geometry and non-linear second-order differential equations" [12], it is sometimes possible to decrease one dimension and to work on the contact space J^1N. The idea is to define, for any differential form $\omega \in \Omega^n(J^1N)$, a second-order differential operator $\Delta_\omega : C^\infty(N) \to \Omega^n(N)$ acting according to the rule

$$\Delta_\omega(f) = j_1(f)^*\omega,$$

where $j_1(f) : N \to J^1N$ is the section corresponding to the function f.

The differential equations of the form $\Delta_\omega = 0$ are said to be of Monge–Ampère type because of their "hessian-like" non-linearity. Despite its very simple description, this classical class of differential equations arouses much interest due to its appearance in different problems of geometry or mathematical physics. We refer the reader to the very rich book *Contact Geometry and Non-linear Differential Equations* [10] for a complete exposition of the theory and for numerous examples.

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A Monge–Ampère equation $\Delta_\omega = 0$ is said to be symplectic if the Monge–Ampère operator Δ_ω is invariant with respect to the Reeb vector field. In other words, the n-form ω lives actually on the cotangent bundle T^*N, and symplectic geometry takes the place of contact geometry. The Monge–Ampère operator is then defined by

$$\Delta_\omega(f) = (\mathrm{d}f)^*\omega.$$

This partial case is in some sense quite generic because of the beautiful result of Lychagin which says that any Monge–Ampère equation admitting a contact symmetry is equivalent (by a Legendre transform on J^1N) to a symplectic one.

We are interested here in symplectic Monge–Ampère equations in two variables. These equations are written as:

$$A\frac{\partial^2 f}{\partial q_1^2} + 2B\frac{\partial^2 f}{\partial q_1 \partial q_2} + C\frac{\partial^2 f}{\partial q_2^2} + D\left(\frac{\partial^2 f}{\partial q_1^2}\frac{\partial^2 f}{\partial q_2^2} - \left(\frac{\partial^2 f}{\partial q_1 \partial q_2}\right)^2\right) + E = 0, \tag{1}$$

with A, B, C, D and E smooth functions of $(q, \frac{\partial f}{\partial q})$. These equations correspond to a 2-form on $T^*\mathbb{R}^2$, or equivalently to tensors on $T^*\mathbb{R}^2$ using the correspondence

$$\omega(\cdot,\cdot) = \Omega(A\cdot,\cdot),$$

Ω being the symplectic form on T^*N. In the non-degenerate case, the traceless part of this tensor A defines either an almost complex structure or an almost product structure and it is integrable if and only if the corresponding Monge–Ampère equation is equivalent to the Laplace equation or the wave equation. This elegant result of Lychagin and Roubtsov [13] is quite frustrating: which kind of integrable geometry could we define for more general Monge–Ampère equations?

It has been noticed in [4] that such a pair of forms (ω, Ω) defines an almost generalized complex structure, a very rich concept defined recently by Hitchin [8] and developed by Gualtieri [6], which interpolates between complex and symplectic geometry. It is easy to see that this almost generalized complex structure is integrable for a very large class of $2D$ Monge–Ampère equations, the equations of *divergent type*. This observation is the starting point for the approach proposed in this paper: the aim is to present these differential equations as "generalized Laplace equations".

In the first part, we write down this correspondence between Monge–Ampère equations in two variables and four-dimensional generalized complex geometry.

In the second part we study the $\overline{\partial}$ operator associated with a Monge–Ampère equation of divergent type and we show how the corresponding conservation laws and generating functions can be seen as "holomorphic objects".

1. Monge–Ampère equations and Hitchin pairs

In what follows, M is the smooth symplectic space $T^*\mathbb{R}^2$ endowed with the canonical symplectic form Ω. Our point of view is local (in particular we do not make any distinction between closed and exact forms) but most of the results presented here have a global version.

A primitive 2-form is a differential form $\omega \in \Omega^2(M)$ such that $\omega \wedge \Omega = 0$. We denote by $\perp : \Omega^k(M) \to \Omega^{k-2}(M)$ the operator $\theta \mapsto \iota_{X_\Omega}(\theta)$, the bivector X_Ω being the bivector dual to Ω. It is straightforward to check that in dimension 4, a 2-form ω is primitive if and only if $\perp\omega = 0$.

1.1. Monge–Ampère operators

Definition. Let ω be a 2-form on M. A two-dimensional submanifold L is a generalized solution of the equation $\Delta_\omega = 0$ if it is bilagrangian with respect to Ω and ω.

Note that a lagrangian submanifold of $T^*\mathbb{R}^2$ which projects isomorphically on $\mathbb{R}^2$ is a graph of a closed 1-form $\mathrm{d}f : \mathbb{R}^2 \to T^*\mathbb{R}^2$. A generalized solution can be thought of as a smooth patching of classical solutions of the Monge–Ampère equation $\Delta_\omega = 0$ on $\mathbb{R}^2$.

Example 1 (*Laplace Equation*). Consider the $2D$ Laplace equation

$$f_{q_1q_1} + f_{q_2q_2} = 0.$$

It corresponds to the form $\omega = \mathrm{d}q_1 \wedge \mathrm{d}p_2 - \mathrm{d}q_2 \wedge \mathrm{d}p_1$, while the symplectic form is $\Omega = \mathrm{d}q_1 \wedge \mathrm{d}p_1 + \mathrm{d}q_2 \wedge \mathrm{d}p_2$. Introducing the complex coordinates $z_1 = q_1 + \mathrm{i}q_2$ and $z_2 = p_2 + \mathrm{i}p_1$, we get $\omega + \mathrm{i}\Omega = \mathrm{d}z_1 \wedge \mathrm{d}z_2$. Generalized solutions of the $2D$ Laplace equation appear then as the complex curves of $\mathbb{C}^2$.

The following theorem (the so called Hodge–Lepage–Lychagin theorem; see [12]) establishes the 1–1 correspondence between Monge–Ampère operators and primitive 2-forms:

Theorem. (i) *Any 2-form admits the unique decomposition $\omega = \omega_0 + \lambda\omega$, with ω_0 primitive.*
(ii) *If two primitive forms vanish on the same lagrangian subspaces, then there are proportional.*

Remark. A Monge–Ampère operator Δ_ω is therefore uniquely defined by the primitive part ω_0 of ω, since $\lambda\Omega$ vanish on any lagrangian submanifold. The function λ can be chosen arbitrarily.

Let $\omega = \omega_0 + \lambda\Omega$ be a 2-form. We define the tensor A by $\omega = \Omega(A\cdot, \cdot)$. One has $A = A_0 + \lambda Id$ and

$$A_0^2 = -\mathrm{pf}(\omega_0)Id,$$

where the function $\mathrm{pf}(\omega_0)$ is the pfaffian of ω_0 defined by

$$\omega_0 \wedge \omega_0 = \mathrm{pf}(\omega_0)\Omega \wedge \Omega.$$

Therefore,

$$A^2 = 2\lambda A - (\lambda^2 + \mathrm{pf}(\omega_0))Id.$$

The equation $\Delta_\omega = 0$ is said to be elliptic if $\mathrm{pf}(\omega_0) > 0$, hyperbolic if $\mathrm{pf}(\omega_0) < 0$, parabolic if $\mathrm{pf}(\omega_0) = 0$. In the elliptic/hyperbolic case, one can define the tensor

$$J_0 = \frac{A_0}{\sqrt{|\mathrm{pf}(\omega_0)|}}$$

which is either an almost complex structure or an almost product structure.

Theorem (*Lychagin–Roubtsov [13]*). *The following assertions are equivalent:*

(i) *The tensor J_0 is integrable.*
(ii) *The form $\omega_0/\sqrt{|\mathrm{pf}(\omega_0)|}$ is closed.*
(iii) *The Monge–Ampère equation $\Delta_\omega = 0$ is equivalent (with respect to the action of local symplectomorphisms) to the (elliptic) Laplace equation $f_{q_1q_1} + f_{q_2q_2} = 0$ or the (hyperbolic) wave equation $f_{q_1q_1} - f_{q_2q_2} = 0$.*

Let us introduce now the Euler operator and the notion of a Monge–Ampère equation of divergent type (see [12]).

Definition. The Euler operator is the second-order differential operator $\mathcal{E} : \Omega^2(M) \to \Omega^2(M)$ defined by

$$\mathcal{E}(\omega) = \mathrm{d} \perp \mathrm{d}\omega.$$

A Monge–Ampère equation $\Delta_\omega = 0$ is said to be of divergent type if $\mathcal{E}(\omega) = 0$.

Example 2 (*Born–Infeld Equation*). The Born–Infeld equation is

$$(1 - f_t)^2 f_{xx} + 2 f_t f_x f_{tx} - (1 + f_x^2) f_{tt} = 0.$$

The corresponding primitive form is

$$\omega_0 = (1 - p_1^2)\mathrm{d}q_1 \wedge \mathrm{d}p_2 + p_1 p_2(\mathrm{d}q_1 \wedge \mathrm{d}p_1) + (1 + p_2^2)\mathrm{d}q_2 \wedge \mathrm{d}p_1$$

with $q_1 = t$ and $q_2 = x$. A direct computation gives

$$\mathrm{d}\omega_0 = 3(p_1\mathrm{d}p_2 - p_2\mathrm{d}p_1) \wedge \Omega,$$

and then the Born–Infeld equation is not of divergent type.

Example 3 (*Tricomi Equation*). The Tricomi equation is

$$v_{xx} x v_{yy} + \alpha v_x + \beta v_y + \gamma(x, y).$$

The corresponding primitive form is

$$\omega_0 = (\alpha p_1 + \beta p_2 + \gamma(q))\mathrm{d}q_1 \wedge \mathrm{d}q_2 + \mathrm{d}q_1 \wedge \mathrm{d}p_2 - q_2 \mathrm{d}q_2 \wedge \mathrm{d}p_1,$$

with $x = q_1$ and $y = q_2$. Since

$$\mathrm{d}\omega_0 = (-\alpha \mathrm{d}q_2 + \beta \mathrm{d}q_1) \wedge \Omega,$$

we conclude that the Tricomi equation is of divergent type.

Lemma. *A Monge–Ampère equation $\Delta_\omega = 0$ is of divergent type if and only if there exists a function μ on M such that the form $\omega + \mu\Omega$ is closed.*

Proof. Since the exterior product by Ω is an isomorphism from $\Omega^1(M)$ to $\Omega^3(M)$, for any 2-form ω, there exists a 1-form α_ω such that

$$\mathrm{d}\omega = \alpha_\omega \wedge \Omega.$$

Since $\perp(\alpha_\omega \wedge \Omega) = \alpha_\omega$ we deduce that $\mathcal{E}(\omega) = 0$ if and only if $\mathrm{d}\alpha_\omega = 0$, that is $\mathrm{d}(\omega + \mu\Omega) = 0$ with $\mathrm{d}\mu = -\alpha_\omega$. □

Hence, if $\Delta_\omega = 0$ is of divergent type, one can choose ω being closed. The point is that it is not primitive in general.

1.2. Hitchin pairs

Let us denote by T the tangent bundle of M and by T^* its cotangent bundle. The natural indefinite interior product on $T \oplus T^*$ is

$$(X + \xi, Y + \eta) = \frac{1}{2}(\xi(Y) + \eta(X)),$$

and the Courant bracket on sections of $T \oplus T^*$ is

$$[X + \xi, Y + \eta] = [X, Y] + L_X\eta - L_Y\xi - \frac{1}{2}\mathrm{d}(\iota_X\eta - \iota_Y\xi).$$

Definition (*Hitchin [8]*). An almost generalized complex structure is a bundle map $\mathbb{J} : T \oplus T^* \to T \oplus T^*$ satisfying

$$\mathbb{J}^2 = -1,$$

and

$$(\mathbb{J}\cdot, \cdot) = -(\cdot, \mathbb{J}\cdot).$$

Such an almost generalized complex structure is said to be integrable if the spaces of sections of its two eigenspaces are closed under the Courant bracket.

The standard examples are

$$\mathbb{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$$

and

$$\mathbb{J}_2 = \begin{pmatrix} 0 & \Omega^{-1} \\ -\Omega & 0 \end{pmatrix}$$

with J a complex structure and Ω a symplectic form.

Lemma (*Crainic [4]*). *Let Ω be a symplectic form and ω any 2-form. Define the tensor A by $\omega = \Omega(A\cdot,\cdot)$ and the form $\tilde{\omega}$ by $\tilde{\omega} = -\Omega(1 + A^2\cdot,\cdot)$.*

The almost generalized complex structure

$$\mathbb{J} = \begin{pmatrix} A & \Omega^{-1} \\ \tilde{\omega} & -A^* \end{pmatrix} \tag{2}$$

is integrable if and only if ω is closed. Such a pair (ω, Ω) with $\mathrm{d}\omega = 0$ is called a Hitchin pair.

We get then immediately the following:

Proposition 1. *To any two-dimensional symplectic Monge–Ampère equation of divergent type $\Delta_\omega = 0$ there corresponds a Hitchin pair (ω, Ω) and therefore a four-dimensional generalized complex structure.*

Remark. Let $L^2 \subset M^4$ be a two-dimensional submanifold. Let $T_L \subset T$ be its tangent bundle and $T_L^0 \subset T^*$ its annihilator. L is a generalized complex submanifold (according to the terminology of [6]) or a generalized lagrangian submanifold (according to the terminology of [2]) if $T_L \oplus T_L^0$ is closed under $\mathbb{J}$. When $\mathbb{J}$ is defined by (2), this is equivalent to saying that L is lagrangian with respect to Ω and closed under A, that is, L is a generalized solution of $\Delta_\omega = 0$.

1.3. Systems of first-order partial differential equations

On $2n$-dimensional manifold, a generalized complex structure is written as

$$\mathbb{J} = \begin{pmatrix} A & \pi \\ \sigma & -A^* \end{pmatrix}$$

with different relations detailed in [4] between the tensor A, the bivector π and the 2-form σ, the most outstanding being $[\pi, \pi] = 0$, that is π is a Poisson bivector.

In [4], a generalized complex structure is said to be non-degenerate if the Poisson bivector π is non-degenerate, that is, if the two eigenspaces $E = \mathrm{Ker}(\mathbb{J} - \mathrm{i})$ and $\overline{E} = \mathrm{Ker}(\mathbb{J} + \mathrm{i})$ are transverse to T^*. This leads to our symplectic form $\Omega = \pi^{-1}$ and to our 2-form $\omega = \Omega(A\cdot,\cdot)$.

One could also take the dual point of view and study generalized complex structure transverse to T. In this situation, the eigenspace E is written as

$$E = \left\{\xi + \iota_\xi P, \xi \in T^* \otimes \mathbb{C}y\right\},$$

with $P = \pi + \mathrm{i}\Pi$ a complex bivector. This space defines a generalized complex structure if and only if it is a Dirac subbundle of $(T \oplus T^*) \otimes \mathbb{C}$ and if it is transverse to its conjugate $\overline{E}$. According to the Maurer–Cartan type equation described in the famous paper *Manin triples for Lie bialgebroids* [11], the first condition is

$$[\pi + \mathrm{i}\Pi, \pi + \mathrm{i}\Pi] = 0.$$

The second condition says that Π is non-degenerate.

Hence, we obtain an analog of Crainic's result:

Definition. A Hitchin pair of bivectors is a pair consisting of two bivectors π and Π, Π being non-degenerate, and satisfying

$$\begin{cases} [\Pi, \Pi] = [\pi, \pi] \\ [\Pi, \pi] = 0. \end{cases} \tag{3}$$

Proposition 2. *There is a 1–1 correspondence between generalized complex structure*

$$\mathbb{J} = \begin{pmatrix} A & \pi_A \\ \sigma & -A^* \end{pmatrix}$$

with σ non-degenerate and Hitchin pairs of bivector (π, Π). In this correspondence, we have

$$\begin{cases} \sigma = \Pi^{-1} \\ A = \pi \circ \Pi^{-1} \\ \pi_A = -(1 + A^2)\Pi. \end{cases}$$

Example 4. If $\pi + \mathrm{i}\Pi$ is non-degenerate, it defines a 2-form $\omega + \mathrm{i}\Omega$ which is necessarily closed (this is the complex version of the classical result which says that a non-degenerate Poisson bivector is actually symplectic). We find again a Hitchin pair. So new examples occur only in the degenerate case. Note that $\pi + \mathrm{i}\Pi = (A + \mathrm{i})\Pi$, so $\det(\pi + \mathrm{i}\Pi) = 0$ if and only if $-\mathrm{i}$ is an eigenvalue for A. In dimension 4, this implies that $A^2 = -1$ but this is no longer true in higher dimensions (see for example the classification of a pair of 2-forms on six-dimensional manifolds in [13]). Nevertheless, the case $A^2 = -1$ is interesting in itself. It corresponds to generalized complex structure of the form

$$\mathbb{J} = \begin{pmatrix} J & 0 \\ \sigma & -J^* \end{pmatrix}$$

with J an integrable complex structure and σ a 2-form satisfying $J^*\sigma = -\sigma$ and

$$\mathrm{d}\sigma_J = \mathrm{d}\sigma(J\cdot, \cdot, \cdot) + \mathrm{d}\sigma(\cdot, J\cdot, \cdot) + \mathrm{d}\sigma(\cdot, \cdot, J\cdot),$$

where $\sigma_J = \sigma(J\cdot, \cdot)$ (see [4]). Or equivalently $\sigma + \mathrm{i}\sigma_J$ is a (2, 0)-form satisfying

$$\partial(\sigma + \mathrm{i}\sigma_J) = 0.$$

One typical example of such geometry is the so called hyper-Kähler geometry with torsion which is an elegant generalization of hyper-Kähler geometry [5]. Unlike in the hyper-Kähler case, such geometries are always generated by potentials [1].

Let us consider now a Hitchin pair of bivectors (π, Π) in dimension 4. Since Π is non-degenerate, it defines two 2-forms ω and Ω, which are not necessarily closed, and which are related by the tensor A. A generalized lagrangian surface is a surface closed under A, or equivalently, bilagrangian: $\omega|_L = \Omega|_L = 0$. Locally, L is defined by two functions u and v satisfying a first-order system

$$\begin{cases} a + b\dfrac{\partial u}{\partial x} + c\dfrac{\partial u}{\partial y} + d\dfrac{\partial v}{\partial x} + e\dfrac{\partial v}{\partial y} + f \det J_{u,v} \\ A + B\dfrac{\partial u}{\partial x} + C\dfrac{\partial u}{\partial y} + D\dfrac{\partial v}{\partial x} + E\dfrac{\partial v}{\partial y} + E \det J_{u,v} \end{cases}$$

with

$$J_{u,v} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}.$$

Such a system generalizes both Monge–Ampère equations and Cauchy–Riemann systems and is called a Jacobi system (see [10]).

With the help of Hitchin's formalism, we understand now the integrability condition (3) as a "divergent type" condition for Jacobi equations.

2. The $\overline{\partial}$ operator

Let us fix now a $2D$ symplectic Monge–Ampère equation of divergent type $\Delta_\omega = 0$, the 2-form $\omega = \omega_0 + \lambda\Omega$ being closed. We still denote by $A = A_0 + \lambda$ the associated tensor.

Lemma. *For any* 1*-form* α*, the following relation holds:*

$$\alpha \wedge \omega - B^*\alpha \wedge \Omega = 0 \tag{4}$$

with $B = \lambda - A_0$*.*

Proof. Let $\alpha = \iota_X \Omega$ be a 1-form. Since ω_0 is primitive, we get

$$0 = \iota_X(\omega_0 \wedge \Omega) = (\iota_X \omega_0) \wedge \Omega + (\iota_X \Omega) \wedge \omega_0 = A_0^* \alpha \wedge \Omega + \alpha \wedge \omega_0.$$

Therefore,

$$\alpha \wedge \omega = \alpha \wedge \omega_0 + \lambda \alpha \wedge \Omega = (-A_0 + \lambda)^* \alpha \wedge \Omega. \quad \square$$

We denote by $\mathbb{J}$ the generalized complex structure associated with the Hitchin pair (ω, Ω). We also define

$$\Theta = \omega - \mathrm{i}\Omega$$

and

$$\Phi = \exp(\Theta) = 1 + \Theta + \frac{\Theta^2}{2}.$$

2.1. Decomposition of forms

Using the tensor $\mathbb{J}$, Gualtieri defines a decomposition

$$\Lambda^*(T^*) \otimes \mathbb{C} = U_2 \oplus U_{-1} \oplus U_0 \oplus U_1 \oplus U_2$$

which generalizes the Dolbeault decomposition for a complex structure [6].

Let us introduce some notation to help us understand this decomposition. The space $T \oplus T^*$ acts on $\Lambda^*(T^*)$ via

$$\rho(X + \xi)(\theta) = \iota_X \theta + \xi \wedge \theta,$$

and this action extends to an isomorphism (the standard spin representation) between the Clifford algebra $CL(T \oplus T^*)$ and the space of linear endomorphisms $\mathrm{End}(\Lambda^*(T^*))$.

Remark. With this notation, the eigenspace $E = \mathrm{Ker}(\mathbb{J} - \mathrm{i})$ is also defined by

$$E = \left\{ X + \xi \in T \oplus T^*, \rho(X + \xi)(\Phi) = 0 \right\}.$$

Definition. The space U_k is defined by

$$U_k = \rho\left(\Lambda^{2-k}\overline{E}\right)(\Phi).$$

Note that $\mathbb{J}$ identified with the 2-form $(\mathbb{J}\cdot, \cdot)$ lives in $\Lambda^2(T \oplus T^*) \subset CL(T \oplus T^*)$. We get then an infinitesimal action of $\mathbb{J}$ on $\Lambda^*(T^*)$.

Lemma (*Gualtieri*). *U_k is the ik eigenspace of $\mathbb{J}$.*

Remark. We see then immediately that $U_{-k} = \overline{U_k}$, since $\mathbb{J}$ is a real tensor.

Proposition 3. (i) $U_2 = \mathbb{C}\Phi$.
(ii) $U_1 = \{\alpha \wedge \Phi, \alpha \in \Lambda^1(T^*) \otimes \mathbb{C}\}$.
(iii) $U_0 = \{(\theta - \frac{\mathrm{i}}{2} \perp \theta) \wedge \Phi, \theta \in \Lambda^2(T^*) \otimes \mathbb{C}\}$.

Proof. The eigenspace $\overline{E}$ is

$$\overline{E} = \{X - \iota_X \overline{\Theta}, X \in T \otimes \mathbb{C}\}.$$

Now,

$$\rho(X - \iota_X \overline{\Theta})(\Phi) = \iota_X \Theta + \iota_X \Theta \wedge \Theta - \iota_X \overline{\Theta} - \iota_X \overline{\Theta} \wedge \Theta = \iota_X(\Theta - \overline{\Theta}) \wedge (1 + \Theta).$$

Since $\Theta - \overline{\Theta} = -2\mathrm{i}\Omega$ and $X \mapsto \iota_X \Omega$ is an isomorphism between T and T^*, we get then the description of U_1.

Choose now two complex vectors X and Y and define $\alpha = \iota_X \Omega$ and $\beta = \iota_Y \Omega$:

$$\begin{aligned}
&\rho((X - \iota_X \overline{\Theta}) \wedge (Y - \iota_Y \overline{\Theta}))(\Phi) \\
&\quad = \rho\left(X - \iota_X \overline{\Theta}\right)(-2\mathrm{i}\beta \wedge \Phi) \\
&\quad = -2\mathrm{i}\rho\left(X - \iota_X \overline{\Theta}\right)(\beta + \beta \wedge \Theta) \\
&\quad = -2\mathrm{i}\left(\beta(X)(1+\Theta) - \beta \wedge \iota_X \Theta - \iota_X \overline{\Theta} \wedge \beta - \iota_X \overline{\Theta} \wedge \beta \wedge \Theta\right) \\
&\quad = -2\mathrm{i}\left(\beta(X)(1+\Theta) + \iota_X(\Theta - \overline{\Theta}) \wedge \beta \wedge (1+\Theta) - \iota_X \Theta \wedge \beta \wedge \Theta\right) \\
&\quad = -2\mathrm{i}\left(\beta(X)(1+\Theta) - 2\mathrm{i}\alpha \wedge \beta \wedge (1+\Theta) + \beta \wedge \iota_X \frac{\Theta^2}{2}\right).
\end{aligned}$$

Moreover, since $\beta \wedge \Theta^2 = 0$, we have $\beta(X)\Theta^2 = \beta \wedge \iota_X \Theta^2$ and then

$$\rho\left((X - \iota_X \overline{\Theta}) \wedge (Y - \iota_Y \overline{\Theta})\right)(\Phi) = -2\mathrm{i}(\beta(X) - 2\mathrm{i}\alpha \wedge \beta) \wedge \Phi.$$

But $\perp (\alpha \wedge \beta) = -\beta(X) = \alpha(Y)$. We obtain then the description of U_0. □

The next proposition describes the space $U_0^{\mathbb{R}}$ of *real* forms in U_0. It is a direct consequence of the proposition above.

Proposition 4. *Let Λ_0^2 be the space of (real) primitive 2-forms. Then*

$$U_0^{\mathbb{R}} = \{[\theta + a(\mathrm{i}\Omega + 1)] \wedge \Phi,\ \theta \in \Lambda_0^2 \text{ and } a \in \mathbb{R}\}.$$

Remark. We have actually

$$(\Lambda^1 \oplus \Lambda^3) \otimes \mathbb{C} = U_{-1} \oplus U_1$$

and

$$(\Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4) \otimes \mathbb{C} = U_{-2} \oplus U_0 \oplus U_2.$$

For example, the decomposition of a 1-form $\alpha \in \Lambda^1(T^*)$ is

$$\alpha = \frac{\alpha - \mathrm{i}B\alpha}{2} \wedge \Phi + \frac{\alpha + \mathrm{i}B\alpha}{2} \wedge \overline{\Phi}.$$

This decomposition is a pointwise decomposition. Denote now by $\mathcal{U}_k$ the space of smooth sections of the bundle U_k. The Gualtieri decomposition is now

$$\Omega^*(M) \otimes \mathbb{C} = \mathcal{U}_{-2} \oplus \mathcal{U}_{-1} \oplus \mathcal{U}_0 \oplus \mathcal{U}_1 \oplus \mathcal{U}_2.$$

Definition. The operator $\overline{\partial} : \mathcal{U}_k \to \mathcal{U}_{k+1}$ is simply $\overline{\partial} = \pi_{k+1} \circ d$.

The next theorem is completely analogous to the corresponding statement involving an almost complex structure and the Dolbeault operator $\overline{\partial}$.

Theorem (*Gualtieri [6]*). *The almost generalized complex structure $\mathbb{J}$ is integrable if and only if*

$$\mathrm{d} = \partial + \overline{\partial}.$$

Example 5. Let $\alpha \in \Omega^1(M)$ be a 1-form. From $\mathrm{d}(\alpha \wedge \Phi) = \mathrm{d}\alpha \wedge \Phi$ we get

$$\begin{cases} \overline{\partial}(\alpha \wedge \Phi) = \dfrac{\mathrm{i}}{2}(\perp \mathrm{d}\alpha)\Phi \\ \partial(\alpha \wedge \Phi) = \left(\mathrm{d}\alpha - \dfrac{\mathrm{i}}{2} \perp \mathrm{d}\alpha\right) \wedge \Phi. \end{cases}$$

It is worth mentioning that one can also define the real differential operator $d^{\mathbb{J}} = [d, \mathbb{J}]$, or equivalently (see [3])

$$d^{\mathbb{J}} = -i(\partial - \overline{\partial}).$$

Remark. Cavalcanti establishes in [3], for the particular case $\omega = 0$, an isomorphism $\Xi : \Omega^*(M) \otimes \mathbb{C} \to \Omega^*(M) \otimes \mathbb{C}$ satisfying

$$\Xi(d\theta) = \partial \Xi(\theta), \qquad \Xi(\delta\theta) = \overline{\partial} \Xi(\theta)$$

with $\delta = [d, \perp]$ the symplectic codifferential. Since $d\delta$ is the Euler operator, Monge–Ampère equations of divergent type are written as $\Delta_\omega = 0$ with $\Xi(\omega)$ pluriharmonic on the generalized complex manifold $\left(M^4, \exp(i\Omega)\right)$.

2.2. Conservation laws and generating functions

The notion of conservation laws is a natural generalization to partial differential equations of the notion of first integrals.

A 1-form α is a conservation law for the equation $\Delta_\omega = 0$ if the restriction of α to any generalized solution is closed. Note that conservations laws are actually well defined up closed forms.

Example 6. Let us consider the Laplace equation and the complex structure J associated with it. The 2-form $d\alpha$ vanishes on any complex curve if and only if $[d\alpha]_{1,1} = 0$, that is

$$\overline{\partial}\alpha_{1,0} + \partial\alpha_{0,1} = 0$$

or equivalently

$$\overline{\partial}\alpha_{1,0} = \overline{\partial}\partial\psi$$

for some real function ψ. (Here $\overline{\partial}$ is the usual Dolbeault operator defined by the integrable complex structure J.) We deduce that $\alpha - d\psi = \beta_{1,0} + \beta_{0,1}$ with $\beta_{1,0} = \alpha_{1,0} - \partial\psi$ a holomorphic (1, 0)-form.

Hence, the conservation laws of the $2D$ Laplace equation are (up to exact forms) real parts of (1, 0)-holomorphic forms.

According to the Hodge–Lepage–Lychagin theorem, α is a conservation law if and only if there exist two functions f and g such that $d\alpha = f\omega + g\Omega$. The function f is called a generating function of the Monge–Ampère equation $\Delta_\omega = 0$. By analogy with the Laplace equation, we will say that the function g is the conjugate function of the generating function f.

Lemma. *A function f is a generating function if and only if*

$$dBdf = 0.$$

Proof. f is a generating function if and only if there exists a function g such that

$$0 = d(f\omega + g\Omega) = df \wedge \omega + dg \wedge \Omega = (dg + Bdf) \wedge \Omega,$$

and therefore g exists if and only if $Bdf = 0$. □

Corollary. *If f is a generating function and g is its conjugate then for any $c \in \mathbb{C}$, $L_c = (f + ig)^{-1}(c)$ is a generalized solution of the Monge–Ampère equation $\Delta_\omega = 0$.*

Proof. The tangent space $T_a L_c$ is generated by the hamiltonian vector fields X_f and X_g. Since

$$\Omega(BX_f, Y) = \Omega(X_f, BY) = df(BY) = Bdf(Y) = dg(Y),$$

we deduce that $X_g = BX_f$ and therefore L_c is closed under $B = \lambda - A_0$. L_c is then closed under A_0 and so bilagrangian with respect to Ω and ω. □

Example 7. A generating function of the $2D$ Laplace equation satisfies $dJdf = 0$, and hence it is the real part of a holomorphic function.

The above lemma has a nice interpretation in the Hitchin/Gualtieri formalism:

Proposition 5. *A function f is a generating function of the Monge–Ampère equation $\Delta_\omega = 0$ if and only if f is a pluriharmonic function on the generalized complex manifold $(M^4, \exp(\omega - i\Omega))$, that is*

$$\partial\overline{\partial} f = 0.$$

Proof. The spaces U_1 and U_{-1} are respectively the i and $-$i eigenspaces for the infinitesimal action of $\mathbb{J}$. So

$$\begin{aligned}\mathbb{J}\mathrm{d}f &= \mathbb{J}\left(\frac{\mathrm{d}f - \mathrm{i}B\mathrm{d}f}{2} \wedge \Phi + \frac{\mathrm{d}f + \mathrm{i}B\mathrm{d}f}{2} \wedge \overline{\Phi}\right)\\ &= \mathrm{i}\left(\frac{\mathrm{d}f - \mathrm{i}B\mathrm{d}f}{2} \wedge \Phi - \frac{\mathrm{d}f + \mathrm{i}B\mathrm{d}f}{2} \wedge \overline{\Phi}\right)\\ &= B\mathrm{d}f + (B^2+1)\mathrm{d}f \wedge \Omega.\end{aligned}$$

Moreover,

$$\mathrm{d}\left(\left(B^2+1\right)\mathrm{d}f \wedge \Omega\right) = \mathrm{d}(B^2\mathrm{d}f \wedge \Omega) = \mathrm{d}(B\mathrm{d}f \wedge \omega) = (\mathrm{d}B\mathrm{d}f) \wedge \omega.$$

We deduce that $\mathrm{d}\mathbb{J}\mathrm{d}f = 0$ if and only if $\mathrm{d}B\mathrm{d}f = 0$. Since $\mathrm{d}\mathbb{J}\mathrm{d}f = 2\mathrm{i}\partial\overline{\partial} f$, the proposition is proved. □

Decompose the function f as $f = f_{-2} + f_0 + f_2$. Since $\partial f_{-2} = 0$ and $\overline{\partial} f_2 = 0$, f is pluriharmonic if and only if f_0 is so. Assume that the $\partial\overline{\partial}$ lemma holds (see [3] and [7]). Then there exists $\psi \in \mathcal{U}_1$ such that

$$\overline{\partial} f_0 = \overline{\partial}\partial\psi.$$

Define then $G_0 \in \mathcal{U}_0$ by $G_0 = \mathrm{i}(\partial\psi - \overline{\partial\psi})$. We obtain

$$\overline{\partial}(f_0 + \mathrm{i}G_0) = 0$$

and f_0 appears as the real part of an "holomorphic object". Nevertheless, this assumption is not really clear. Does the $\partial\overline{\partial}$ lemma always hold locally?

The following proposition gives an alternative "holomorphic object" when the closed form ω is primitive (that is $\lambda = 0$).

Proposition 6. *Assume that the closed form ω is primitive and consider the real forms $U = \omega \wedge \Phi$ and $V = (\mathrm{i}\Omega + 1) \wedge \Phi$.*

A function f is a generating function of the Monge–Ampère equation $\Delta_\omega = 0$ with conjugate function g if and only if

$$\overline{\partial}(fU - \mathrm{i}gV) = 0.$$

Proof. According to Proposition 4, the closed forms U and V live in $\mathcal{U}_0^{\mathbb{R}}$. Therefore, $\mathrm{d}^{\mathbb{J}}(fU) = -\mathbb{J}\mathrm{d}(fU)$ and $\mathrm{d}^{\mathbb{J}}(gV) = -\mathbb{J}\mathrm{d}(gV)$. Since $\mathbb{J}^2 = -1$ on $U_{-1} \oplus U_1$, we get

$$2\overline{\partial}(fU - \mathrm{i}gV) = (\mathrm{d} - \mathrm{i}\mathrm{d}^{\mathbb{J}})(fU - \mathrm{i}gV) = (1 + \mathrm{i}\mathbb{J})(\mathrm{d}fU - \mathrm{d}^{\mathbb{J}}gV).$$

But,

$$\mathrm{d}fU = \mathrm{d}f \wedge \omega \wedge \Phi = \mathrm{d}f \wedge \omega,$$

and

$$\begin{aligned}\mathrm{d}^{\mathbb{J}}gV &= -\mathbb{J}\mathrm{d}g \wedge V\\ &= -\mathbb{J}(\mathrm{i}\mathrm{d}g \wedge \Omega + \mathrm{d}g \wedge \Phi)\\ &= -\frac{1}{2}\mathbb{J}(\mathrm{d}g \wedge \Phi + \mathrm{d}g \wedge \overline{\Phi})\\ &= -\frac{\mathrm{i}}{2}(\mathrm{d}g \wedge \Phi - \mathrm{d}g \wedge \overline{\Phi})\\ &= -\mathrm{d}g \wedge \Omega.\end{aligned}$$

We obtain finally

$$2\overline{\partial}(fU - \mathrm{i}gV) = \mathrm{d}f \wedge \omega + \mathrm{d}g \wedge \Omega. \quad \square$$

Example 8 (*Von Karman Equation*). The $2D$ Von Karman equation is

$$v_x v_{xx} - v_{yy} = 0.$$

The corresponding primitive form is

$$\omega = p_1 \mathrm{d}q_2 \wedge \mathrm{d}p_1 + \mathrm{d}q_1 \wedge \mathrm{d}p_2,$$

which is obviously closed. The forms U and V are

$$\begin{cases} U = p_1\mathrm{d}q_2 \wedge \mathrm{d}p_1 + \mathrm{d}q_1 \wedge \mathrm{d}p_2 + 2p_1\mathrm{d}q_1 \wedge \mathrm{d}q_2 \wedge \mathrm{d}p_1 \wedge \mathrm{d}p_2 \\ V = 1 + p_1\mathrm{d}q_2 \wedge \mathrm{d}p_1 + \mathrm{d}q_1 \wedge \mathrm{d}p_2 + (p_1 - 1)\mathrm{d}q_1 \wedge \mathrm{d}q_2 \wedge \mathrm{d}p_1 \wedge \mathrm{d}p_2. \end{cases}$$

2.3. Generalized Kähler partners

Gualtieri has also introduced the notion of generalized Kähler structure. This is a pair of commuting generalized complex structures such that the symmetric product $(\mathbb{J}_1\mathbb{J}_2)$ is positive definite. The remarkable fact in this theory is that such a structure gives for free two integrable complex structures and a compatible metric (see [6]). This theory has been used to construct explicit examples of bihermitian structures on four-dimensional compact manifolds (see [9]).

The idea is that the $+1$ eigenspace V_+ of $\mathbb{J}_1\mathbb{J}_2$ is closed under $\mathbb{J}_1$ and $\mathbb{J}_2$ and that the restriction of $(\cdot,\cdot)$ to it is positive definite. The complex structures and the metric come then from the natural isomorphism $V_+ \to T$.

From our point of view, this approach gives us the possibility of associating with a given partial differential equation, natural integrable complex structures and inner products. Nevertheless, at least for hyperbolic equations, such an inner product should have a signature, and we have maybe to a relax a little bit the definition of generalized Kähler structure:

Definition. Let $\Delta_\omega = 0$ be a $2D$ symplectic Monge–Ampère equation of divergent type and let $\mathbb{J}$ be the generalized complex structure associated with it. We will say that this Monge–Ampère equation admits a generalized Kähler partner if there exists a generalized complex structure $\mathbb{K}$ commuting with $\mathbb{J}$ such that the two eigenspaces of $\mathbb{J}\mathbb{K}$ are transverse to T and T^*.

Note that a powerful tool has been used in [9] to construct such structures:

Lemma (*Hitchin*). *Let* $\exp\beta_1$ *and* $\exp\beta_2$ *be two complex closed forms defining generalized complex structures* $\mathbb{J}_1$ *and* $\mathbb{J}_2$ *on a four-dimensional manifold. Suppose that*

$$(\beta_1 - \beta_2)^2 = 0 = (\beta_1 - \overline{\beta_2})^2;$$

then $\mathbb{J}_1$ *and* $\mathbb{J}_2$ *commute.*

Let us see now for a particular case how one can use this tool. Consider an elliptic Monge–Ampère equation $\Delta_\omega = 0$ with $\mathrm{d}\omega = 0$ and $\Omega \wedge \omega = 0$. Assume moreover that there exists a closed 2-form Θ such that

$$\Omega \wedge \Theta = \omega \wedge \Theta = 0$$

and

$$4\omega = \Omega^2 + \Theta^2.$$

Note that $\exp(\omega - \mathrm{i}\Omega)$ and $\exp(-\omega - \mathrm{i}\Theta)$ satisfy the conditions of the above lemma. We suppose also that $\Theta^2 = \lambda^2\Omega$ with λ a non-vanishing function. This implies that $\omega^2 = \mu^2\Omega^2$ with

$$\mu = \frac{\sqrt{1+\lambda^2}}{2}.$$

The triple (ω, Ω, Θ) defines a metric G and an almost hypercomplex structure (I, J, K) such that

$$\omega = \mu G(I\cdot, \cdot), \qquad \Omega = G(J\cdot, \cdot), \qquad \Theta = \lambda G(K\cdot, \cdot).$$

Define now the two almost complex structures

$$I_+ = \frac{K + \lambda J}{\mu}, \quad I_- = \frac{K - \lambda J}{\mu}.$$

From

$$\omega = \frac{\Omega + \Theta}{2}(I_-\cdot, \cdot)$$

and

$$\omega = \frac{\Omega - \Theta}{2}(I_+\cdot, \cdot)$$

we deduce that I_+ and I_- are integrable.

Lemma. *A function g is the conjugate of a generating function f of the Monge–Ampère equation $\Delta_\omega = 0$ if and only if*

$$\mathrm{d}I_+\mathrm{d}g = -\mathrm{d}I_-\mathrm{d}g.$$

Proof. f is a generating function with conjugate g if and only if

$$0 = \mathrm{d}f \wedge \omega + \mathrm{d}g \wedge \Omega = (-\mu K\mathrm{d}f + \mathrm{d}g) \wedge \Omega,$$

that is if and only if $\mathrm{d}\frac{K}{\mu}\mathrm{d}g = 0$. □

Example 9. Consider again the Von Karman equation

$$v_x v_{xx} - v_{yy} = 0$$

with corresponding primitive and closed form

$$\omega = p_1\mathrm{d}q_2 \wedge \mathrm{d}p_1 + \mathrm{d}q_1 \wedge \mathrm{d}p_2.$$

Define then Θ by

$$\Theta = \mathrm{d}p_1 \wedge \mathrm{d}p_2 + (1 + 4p_1)\mathrm{d}q_1 \wedge \mathrm{d}q_2.$$

With the triple (ω, Ω, Θ) we construct I_+ and I_- defined by

$$I_+ = \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 & 0 \\ -1/p_1 & 0 & 0 & -1/p_1 \\ -(1+4p_1)/p_1 & 0 & 0 & -1/p_1 \\ 0 & 1+4p_1 & -1 & 0 \end{pmatrix}$$

$$I_- = \frac{1}{2}\begin{pmatrix} 0 & -1 & -1 & 0 \\ -1/p_1 & 0 & 0 & 1/p_1 \\ (1+4p_1)/p_1 & 0 & 0 & -1/p_1 \\ 0 & -(1+4p_1) & -1 & 0 \end{pmatrix}.$$

It is worth mentioning that I_+ and I_- are well defined for all $p_1 \neq 0$. But the metric G is positive definite only for $p_1 < -\frac{1}{4}$.

Remark. It would be very interesting to understand the behaviour of generating functions and generalized solutions of such Monge–Ampère equations with respect to the Gualtieri metric. In particular, Gualtieri has introduced a scheming generalized Laplacian dd* + d*d (see [7]) and to know whether generating functions (which are pluriharmonic as we have seen) are actually harmonic would give important information on the global nature of the solutions. This will be the object of further investigations.

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